ON SOME NONPARAMETRIC TESTS FOR PROFILE ANALYSIS OF SEVERAL MULTIVARIATE SAMPLES

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Summary

For the profile analysis of several multivariate samples in a nonparametric framework without assuming normality, a certain appropriate hypothesis of parallelism of population profiles is formulated. A class of test criteria is obtained to test such an hypothesis. The overall χ^2-statistic arising from differences among populations is partitioned into two components--the first due to the "interaction" between populations and variables, and the remainder due to the "pure main effects" from the populations. Some theoretical properties of the criteria are established and, finally, simulation studies are carried out to investigate the performance of these criteria for small or moderate size samples.

Introduction

Let $\underset{\sim}{X}'_{ij} = (X_{ij}^{(1)}, X_{ij}^{(2)}, \ldots, X_{ij}^{(p)})$, $j=1,2,\ldots,n_i$ be independent random vectors from the i-th population with nonsingular continuous c.d.f. F_i. Assume that we have such independent samples from k populations for $i=1,2,\ldots,k$ with a total sample of size $N = \Sigma_i n_i$ on p variables. In the parametric framework it is usually assumed that the i-th population is p-variate normal with mean $\underset{\sim}{\mu}' = (\mu_i^{(1)},\ldots,\mu_i^{(p)})$ and common nonsingular covariance matrix $\underset{\sim}{\Sigma}$. The hypothesis of homogeneity is then

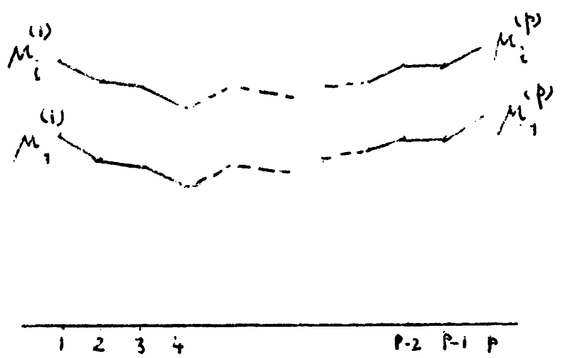
$$H_0: \ \underset{\sim}{\mu}_1 = \underset{\sim}{\mu}_2 = \cdots = \underset{\sim}{\mu}_k,$$

while the hypothesis of parallelism of population profiles is usually formulated as

$$H_1: \ \mu_i^{(1)}-\mu_1^{(1)} = \cdots = \mu_i^{(p)}-\mu_1^{(p)}. \quad i=2,\ldots,k.$$

The formulation H_1 arises naturally by considering population profiles obtained by plotting means $\mu_i^{(\alpha)}$ against variable $\alpha=1,\ldots,p$ for each i.

Fig. 1: Population Profiles in terms of Means



The corresponding sample profiles are obtained on replacing population means by sample means. The appropriate test criteria for H_0 and H_1 respectively are available in standard texts (see, e.g., [5]).

It is desirable to construct suitable nonparametric analogs in order to discard the stringent assumption of p-variate normality, especially in situations where we can observe ordinal data without precise numerical measurements X's. Nonparametric tests are already available in the statistical literature (see, e.g., [2], [6], [7]) for the homogeneity hypothesis

$$H_0^*: \ F_1=F_2=\cdots=F_k,$$

which is the obvious analog of H_0 in the nonparametric case. In this paper a suitable nonparametric analog of H_1, viz. H_1^*, is first formulated and, then, asymptotic chi-square criteria are offered to test H_1^*.

Preliminaries

The nonparametric tests of H_0^* presented independently by Bhapkar [2] and Suguira [7] are based on the technique of generalized U-statistics. Such tests were developed initially for the univariate case by Bhapkar ([1]).

In the multivariate case the generalized U-statistic $U_i^{(\alpha)}$ corresponding to a function $\phi_i^{(\alpha)}$ of k arguments is defined by

$$U_i^{(\alpha)} = \frac{1}{\prod_{j=1}^{k} n_j} \sum_{t_1=1}^{n_1} \cdots \sum_{t_k=1}^{n_k} \phi_i^{(\alpha)}(\underset{\sim}{X}_{1t_1},\ldots,\underset{\sim}{X}_{kt_k}), \quad (2)$$

$\alpha = 1,\ldots,p$ and $i = 1,\ldots,k$. Let

$$\underset{\sim}{U}'_1 = (U_i^{(1)},\ldots,U_i^{(p)})$$

and

$$\underset{\approx}{U}' = (\underset{\sim}{U}'_1,\ldots,\underset{\sim}{U}'_k).$$

We assume that $\phi_i^{(\alpha)}$ is a specific rank function, say ϕ, comparing the α-th components of the i-th argument against the other k-1. Thus, we assume

$$\phi_i^{(\alpha)}(\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_k) = \phi(r_i^{(\alpha)}), \quad (3)$$

where $r_i^{(\alpha)}$ is the rank of $x_i^{(\alpha)}$ among $\{x_j^{(\alpha)}, j=1,\ldots,k\}$. In view of continuity assumption, with probability one there are no ties. Note that the functions considered by Bhapkar [2] and Suguira [7] are special cases of functions satisfying (3).

Let $\underset{\sim}{F}' = (F_1,\ldots,F_k)$ and define

$\eta^{(\alpha)}(\underset{\sim}{F})=E(U_i^{(\alpha)})=E\phi_i^{(\alpha)}(\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_k)$, where $\underset{\sim}{X}_i$'s represent independent random vectors with c.d.f. F_i's respectively. Then we have

$$\eta_i^{(\alpha)}(\underset{\sim}{F}) = \sum_{j=1}^{k} \phi(j)P[R_i^{(\alpha)} = j] = \sum_{j=1}^{k} \phi(j)\nu_{ij}^{(\alpha)}(\underset{\sim}{F}); \quad (4)$$

here $R_i^{(\alpha)}$ is the rank of $X_i^{(\alpha)}$ among $\{X_j^{(\alpha)}, j=1,\ldots,k\}$ and

$$\nu_{ij}^{(\alpha)}(\underset{\sim}{F}) = P[R_i^{(\alpha)} = j], \quad (5)$$

with the probabilities computed under $\underset{\sim}{F}$.

Suppressing $\underset{\sim}{F}$ for the moment, let

$$\underset{\sim}{\eta}_i' = (\eta_i^{(1)},\ldots,\eta_i^{(p)}),\ \underset{\sim}{\eta}' = (\underset{\sim}{\eta}_1',\ldots,\underset{\sim}{\eta}_k').$$

Note that under H_0^*, $\nu_{ij}^{(\alpha)} = 1/k$ for all $\alpha = 1,\ldots,p$, and $i,j = 1,\ldots,k$, so that $\underset{\sim}{\eta}(\underset{\sim}{F}) = \phi\underset{\sim}{j}$, where $\underset{\sim}{j}$ is a column-vector of appropriate order with all elements 1 and

$$\phi = \frac{1}{k}\sum_{j=1}^{k}\phi(j). \quad (6)$$

Now it is known that if $n_i \to \infty$ in such a way that $n_i/N \to p_i$, where $N = \Sigma_i n_i$, $0 < p_i < 1$, $i=1,\ldots,k$, then

$$E(\underset{\sim}{U}_n) = \underset{\sim}{\eta}(\underset{\sim}{F}),\quad V(\underset{\sim}{U}_n) = \frac{1}{N}\underset{\sim}{T}(\underset{\sim}{F}) + O(N^{-3/2}) \quad (7)$$

and

$$N^{1/2}(\underset{\sim}{U}_n - \underset{\sim}{\eta}(\underset{\sim}{F})) \xrightarrow{\mathcal{L}} \mathcal{N}(\underset{\sim}{0}, \underset{\sim}{T}(\underset{\sim}{F})),$$

for any $\underset{\sim}{F}$. Here the subscript $\underset{\sim}{n}$ denotes the vector of sample sizes on which $\underset{\sim}{U}$ is based, V denotes the covariance matrix, $\mathcal{L}$ denotes convergence in distribution, $\mathcal{N}$ the normal vector of appropriate dimensions.

It was shown in [2] that under $H_0^*(1)$,

$$\underset{\sim}{\eta}(F) = \phi\underset{\sim}{j},\ \underset{\sim}{T}(F) = \underset{\sim}{\Sigma}\otimes\rho(F), \quad (8)$$

where $\underset{\sim}{A}\otimes\underset{\sim}{B} = [a_{ij}\underset{\sim}{B}]$, and $\underset{\sim}{\Sigma} = [\sigma_{ij}]$ is given by

$$\underset{\sim}{\Sigma} = \frac{\mu}{(k-1)^2}\{q\underset{\sim}{J} + k^2\underset{\sim}{\Delta} - k\underset{\sim}{q}\underset{\sim}{j}' - k\underset{\sim}{j}\underset{\sim}{q}'\} \quad (9)$$

with $\underset{\sim}{J} = [1]$, $\underset{\sim}{\Delta}$ diagonal $(p_i^{-1}, i=1,\ldots,k)$, $q = \Sigma_i p_i^{-1}$ and $\underset{\sim}{q}' = (p_1^{-1},\ldots,p_k^{-1})$. Also $\underset{\sim}{\rho}$ is a matrix of correlation coefficients $\rho_{\alpha\beta}$ between $\phi_i^{(\alpha)}(\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_k)$ and $\phi_i^{(\beta)}(\underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_k)$, where $\underset{\sim}{X}$'s and Y's are independent with common c.d.f. F except that $\underset{\sim}{X}_i = \underset{\sim}{Y}_i$, and

$$\mu = E[\psi^2(X_i^{(\alpha)})] - [E(\psi(X_i^{(\alpha)})]^2 \quad (10)$$

where

$$\psi(x_i^{(\alpha)}) = E\{\phi_i^{(\alpha)}(\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_k)|\underset{\sim}{X}_i = \underset{\sim}{x}_i\}.$$

It can be shown that in view of condition (3) μ does not depend on the common F under H_0^*; however it does depend on the function ϕ. Explicit values of μ are given in [2] for some specific functions ϕ.

Also it has been shown in [2] that if, under H_0^*, the common F is nonsingular (in the sense that the whole probability mass is not contained in any lower-dimensional space) then $\underset{\sim}{\rho}(F)$ is nonsingular. Then the matrix $\hat{\underset{\sim}{\rho}}$ of consistent estimators is also nonsingular with probability approaching one as all $n_i \to \infty$.

It was shown in [2] that

$$T_0 = \frac{N(k-1)^2}{\mu k^2}\sum_{i=1}^{k} p_i(\underset{\sim}{U}_i - \bar{\underset{\sim}{U}})\hat{\underset{\sim}{\rho}}^{-1}(\underset{\sim}{U}_i - \bar{\underset{\sim}{U}}), \quad (11)$$

where $p_i = n_i/N$ and $\bar{\underset{\sim}{U}} = \Sigma_i p_i \underset{\sim}{U}_i$, has a limiting $\chi^2(p(k-1))$ distribution under H_0^*. Explicit statistics denoted by V,B,L and W were offered as possible nonparametric test criteria (for the hypothesis H_0^*) corresponding to (i) $\phi_V(r)=1$, if $r=1$, and 0 otherwise, (ii) $\phi_B(r) = 1$, if $r=k$, and 0 otherwise, (iii) $\phi_L(1) = -1$, $\phi_L(k) = 1$ and $\phi_L(r) = 0$ otherwise, and (iv) $\phi_W(r) = r$.

Suguira [7] considered the class of functions

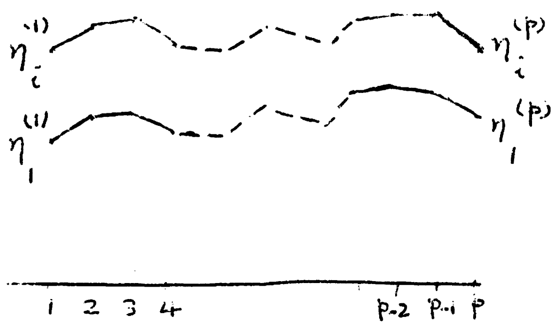
$$\phi_i^{(\alpha)}(x_1,\ldots,x_k) = \frac{(j-1)_r}{(k-1)_r} - \frac{(k-j)_s}{(k-1)_s}, \quad (12)$$

where j is the rank of $x_i^{(\alpha)}$ among $\{x_\ell^{(\alpha)}, \ell=1,\ldots,k\}$, and $(a)_r = a!/(a-r)!$. In view of (3), this function can be expressed as a member of the class $\{\phi_{r,s}, r,s = 0,1,\ldots,k-1\}$ with $\phi_{r,s}(j)$ denoting the right side of (12). The choices (i) $(0,k-1)$, (ii) $(k-1, 0)$, (iii) $(k-1, k-1)$ and (iv) $(1, 1)$ for (r, s), respectively, are essentially equivalent to ϕ_V, ϕ_B, ϕ_L and ϕ_W respectively. His statistic is essentially the same as (11) except that he uses somewhat different estimates for $\underset{\sim}{\rho}$. We may note here, however, that his estimates are consistent only under H_0^* while those in [2] are valid for any $\underset{\sim}{F}$ and hence the latter are to be preferred.

Nonparametric Parallelism Hypothesis

First we want to formulate an appropriate nonparametric analogue of the hypothesis H_1 of parallelism of profiles. In the parametric case the profiles are defined in terms of population means as in Figure 1. In the more general nonparametric case we can similarly define the population profiles in terms of quantities $\eta_i^{(\alpha)}$ which are the expected values of rank functions ϕ in (3). For each i, $\eta_i^{(\alpha)}$ indicates the relative rank location (for the specific ϕ function used) of the i-th population among the k populations with respect to the component α. Plotting these for various α would give the profile of the i-th population.

Fig. 2: Population Profiles in terms of η's



The corresponding sample profiles can similarly be plotted in terms of statistics $U_i^{(\alpha)}$, $\alpha=1,\ldots,p$ for each $i=1,\ldots,k$.

The obvious way to define the parallelism hypothesis is to require

$$\eta_i^{(1)} = \eta_i^{(2)} = \cdots = \eta_i^{(p)}$$

for each i. However, we note from (4) that η's do depend on the specific function ϕ. It is not desirable to formulate the hypothesis itself oriented towards a particular function ϕ. Rather we would prefer a formulation that works no matter which ϕ function is used. With this point in view, we now give the following definition:

Definition. The populations $F_1, F_2, \ldots, F_k$ are said to have parallel profiles if $\underset{\sim}{F} = (F_1,\ldots F_k)$ satisfies

$$H_1^*: \quad \nu_{ij}^{(1)}(\underset{\sim}{F}) = \cdots = \nu_{ij}^{(p)}(\underset{\sim}{F}),\ i,j = 1,\ldots,k, \tag{13}$$

where $\nu_{ij}^{(\alpha)}(\underset{\sim}{F})$ is defined by (5).

One might wonder whether H_1 and H_1^* are equivalent in some sense under the normality assumption. The answer is no except possibly the special case where the variances $\sigma_{\alpha\alpha}$ of $X_i^{(\alpha)}$ are the same for all $\alpha=1,\ldots,p$. We prove here only the weaker statement:

Lemma. If $\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_k$ are independent $\mathcal{N}(\underset{\sim}{\mu}_i, \underset{\sim}{\Sigma})$, respectively, and the diagonal elements of $\underset{\sim}{\Sigma}$ are equal, then H_1 implies H_1^*.

Proof: Note that

$$\nu_{ij}^{(\alpha)}(\underset{\sim}{F}) = P[R_i^{(\alpha)}=j] = \sum_c P[\text{Each of } \{X_{i\ell}^{(\alpha)},\ \ell=1,\ldots,j-1\} < X_i^{(\alpha)} < \text{Each of } \{X_{im}^{(\alpha)},\ m=j+1,\ldots,k\}]$$

$$= \sum_c P[\text{Each of } \{Y_{i\ell}^{(\alpha)}+\mu_{i\ell}^{(\alpha)}-\mu_i^{(\alpha)}\} < Y_i^{(\alpha)} < \text{Each of } \{Y_{im}^{(\alpha)} + \mu_{im}^{(\alpha)}-\mu_i^{(\alpha)}\}]; \tag{14}$$

here $\sum_c$ denote the sum over $\binom{k-1}{j-1}$ combinations of subscripts $i\ell$, $\ell = 1,\ldots,j-1$ chosen out of k-1 distinct subscripts $i\ell$, $\ell=1,\ldots,k$ (except j) (denoting integers $1,\ldots,k$ except i).

Now $Y_i^{(\alpha)}$ for $\alpha = 1,\ldots,p$ and $i = 1,\ldots,k$ are independent and identical normal variables. If the condition H_1 is satisfied, we see from (14) that $\nu_{ij}^{(\alpha)}(\underset{\sim}{F})$ does not depend on α and hence, then, H_1^* is satisfied.

In fact normality as such is not used at all except for the fact that $\underset{\sim}{\mu}_i$ are location parameters. By using essentially the same argument we have thus proved the

Theorem 1. Suppose $\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_k$ are independent with c.d.f.

$$F_i(\underset{\sim}{x}) = F(\underset{\sim}{x}-\underset{\sim}{\mu}_i), \quad i=1,\ldots,k \tag{15}$$

for some continuous F, and assume that the marginal c.d.f.'s $F^{(\alpha)}$, $\alpha=1,\ldots,p$, of F are identical, then the conditon H_1 implies the condition H_1^*.

Test of H_1^*

In order to test H_1^* we now propose the statistic

$$T_1 = \frac{N(k-1)^2}{\mu k^2} \sum_{i=1}^{k} p_i(\underset{\sim}{U}_i-\bar{\underset{\sim}{U}})'[\hat{\underset{\sim}{\rho}}^{-1}-\hat{\gamma}\hat{\underset{\sim}{\rho}}^{-1}\underset{\sim}{J}\hat{\underset{\sim}{\rho}}^{-1}](\underset{\sim}{U}_i-\bar{\underset{\sim}{U}})$$

$$= T_0 - T_2, \tag{16}$$

where T_0 is the statistic (11) for H_0^*,

$$T_2 = \frac{N(k-1)\hat{\gamma}}{\mu k^2} \sum_{i=1}^{k} p_i(\underset{\sim}{U}_i-\bar{\underset{\sim}{U}})'\hat{\underset{\sim}{\rho}}^{-1}\underset{\sim}{J}\hat{\underset{\sim}{\rho}}^{-1}(\underset{\sim}{U}_i-\bar{\underset{\sim}{U}}), \tag{17}$$

and $\hat{\gamma} = 1/\underset{\sim}{j}'\hat{\underset{\sim}{\rho}}^{-1}\underset{\sim}{j}$. T_1 is to be regarded as a large-sample $\chi^2((p-1)(k-1))$ criterion for H_1^* while T_2 as a $\chi^2(k-1)$ criterion for testing H_0^*, assuming H_1^*, i.e., for testing the 'pure' differences among the populations after eliminating from T_0 the interaction contribution, if any.

It may be noted here that if $\underset{\sim}{P}$ is any (p-1)xp matrix of rank p-1 satisfying $\underset{\sim}{P}\underset{\sim}{j} = \underset{\sim}{0}$, then

$$\underset{\sim}{\rho}^{-1} - \gamma\underset{\sim}{\rho}^{-1}\underset{\sim}{J}\underset{\sim}{\rho}^{-1} = \underset{\sim}{P}'(\underset{\sim}{P}\underset{\sim}{\rho}\underset{\sim}{P}')^{-1}\underset{\sim}{P},$$

where $\underset{\sim}{\rho}$ is a positive definite matrix and $\gamma = 1/\underset{\sim}{j}'\underset{\sim}{\rho}^{-1}\underset{\sim}{j}$. Since $\underset{\sim}{\rho}$ is a non-singular correlation matrix, it is positive definite, and so is $\hat{\rho}$ with probability tending to one as $n_i \to \infty$. Thus, we may also express T_1 as

$$T_1 = \frac{N(k-1)^2}{\mu k^2} \sum_{i=1}^{k} p_i(\underset{\sim}{U}_i-\bar{\underset{\sim}{U}})' \underset{\sim}{P}'(\underset{\sim}{P}\hat{\underset{\sim}{\rho}}\underset{\sim}{P}')^{-1}\underset{\sim}{P}(\underset{\sim}{U}_i-\bar{\underset{\sim}{U}}). \quad (18)$$

It is straightforward to show that, if H_0^* holds, $T_1 \xrightarrow{\mathcal{L}} \chi^2((p-1)(k-1))$ and $T_2 \xrightarrow{\mathcal{L}} \chi^2(k-1)$; this will also follow from Theorem 3 established in the next section. However, what we would like to have if possible is the stated limiting distribution of T_1 under H_1^* alone. This does not seem to be possible by the present approach (and perhaps by any other approach) without discarding the relative simple form of the statistic. Note in (7) that in general the limiting covariance matrix $\underset{\sim}{T}$ is a pk x pk matrix of functionals depending on $\underset{\sim}{F}$. It is only under H_0^* that $\underset{\sim}{T}$ had the structure $\underset{\sim}{\Sigma} \otimes \underset{\sim}{\rho}$, where $\underset{\sim}{\Sigma}$ is known, and now $\underset{\sim}{\rho}$ is a pxp matrix of functionals depending on common F. Discarding the Kornecker product structure would make it necessary to estimate all terms of $\underset{\sim}{T}$, thus making the computation much more involved. However, as we shall show shortly, the use of concept of 'local alternatives' to H_0^* still makes it possible to justify the use of statistic T_1 for testing H_1^*.

We now state here without proof a Theorem which establishes consistency of the T_0, T_1 and T_2 tests for appropriate alternatives. The reader is referred to [3] for further details.

Theorem 2. Let T_0, T_1 and T_2 be defined as (11), (16) and (17) for functions $\phi_i^{(\alpha)}$ satisfying (3). If $n_i \to \infty$ in such a way that $n_i/N \to p_i$, $0 < p_i < 1$, then

(i) $T_0 \overset{P}{\to} \infty$ iff $\underset{\sim}{F} \notin \{\underset{\sim}{F} | \Sigma_j \phi(j)\nu_{ij}^{(\alpha)}(\underset{\sim}{F})$ is independent of i and α, $i = 1,\ldots,k$, $\alpha = 1,\ldots,p\}$,

(ii) $T_1 \overset{P}{\to} \infty$ iff $\underset{\sim}{F} \notin \{\underset{\sim}{F} | \Sigma_j \phi(j)\nu_{ij}^{(\alpha)}(\underset{\sim}{F})$ is independent of $\alpha = 1,\ldots,p$ for each $i = 1,\ldots,k\}$

and, if $\underset{\sim}{\rho}^{-1} = [\rho^{\alpha\beta}]$, then

(iii) $T_2 \overset{P}{\to} \infty$ iff $\underset{\sim}{F} \notin \{\underset{\sim}{F} | \Sigma_j \phi(j) \Sigma_{\alpha,\beta} \rho^{\alpha\beta} \nu_{ij}^{(\alpha)}(\underset{\sim}{F})$ is independent of i$\}$.

Remark. We thus note here that the tests T_0, T_1 designed for H_0^*, H_1^* respectively are consistent only against alternatives to the hypotheses 'effectively' being tested viz.

$$H_{0\phi}^*: \quad \sum_{j=1}^{k} \phi(j)\nu_{ij}^{(\alpha)}(\underset{\sim}{F}) \text{ is independent of i and } \alpha$$

and

$$H_{1\phi}^*: \quad \sum_{j=1}^{k} \phi(j)\nu_{ij}^{(\alpha)}(\underset{\sim}{F}) \text{ is independent of } \alpha,$$

depending on the function ϕ used for T's. Of course this undesirable feature of nonparametric tests is usually unavoidable, e.g., Mann-Whitney test, Sign test, Kruskal-Wallis test all suffer from a similar disadvantage.

Note also that if H_1^* is accepted, i.e. $\nu_{ij}^{(\alpha)}$ is independent of α, then $T_2 \overset{P}{\to} \infty$ unless $\Sigma_j \phi(j)\nu_{ij}(\underset{\sim}{F})$ is independent of i which is precisely the condition for $T_1 \overset{P}{\not\to} \infty$ assuming H_1^*.

Asymptotic Distributions

In the previous Theorem we have found the class of fixed alternatives $\underset{\sim}{F}' = (F_1,\ldots,F_k)$ for which the tests are consistent, i.e., for which the power of the respective test tends to 1 as $n_i \to \infty$. We shall now find the limiting distributions of T_0, T_1 and T_2 under the sequence of Pitman location alternatives

$$H_N: \quad F_{iN}(\underset{\sim}{x}) = F(\underset{\sim}{x} - N^{-1/2}\underset{\sim}{\delta}_i), \quad i=1,\ldots,k \quad (19)$$

where the $\underset{\sim}{\delta}_i$'s are not all equal, and $\Sigma_i \underset{\sim}{\delta}_i = \underset{\sim}{0}$. Let

$$\gamma_i^{(\alpha)}(F) = k\, \delta_i^{(\alpha)} q^{(\alpha)}(\phi, F), \quad \underset{\sim}{\gamma}_i' = (\gamma_i^{(1)},\ldots,\gamma_i^{(p)}),$$

$$\underset{\sim}{\gamma}' = (\gamma_1',\ldots,\gamma_k'), \quad (20)$$

where

$$q^{(\alpha)}(\phi,F) = \sum_{j=1}^{k} \phi(j)\left[\binom{k-2}{j-2} a^{(\alpha)}(j-2,k-j,F) - \binom{k-2}{j-1} a^{(\alpha)}(j-1,k-j-1,F)\right]$$

and

$$a^{(\alpha)}(b,c,F) = \int_{-\infty}^{\infty} [F^{(\alpha)}(y)]^b\, [1-F^{(\alpha)}(y)]^c\, f^{(\alpha)}(y)\, dF^{(\alpha)}(y).$$

The result concerning limiting distribution is now stated here without proof. The reader is referred to [3] for further details.

Theorem 3. Consider the sequence $\{H_N\}$ of distributions $\{\underset{\sim}{F}_N\}$ given by (9) and assume that $F^{(\alpha)}$ is differentiable and has a bounded derivative $f^{(\alpha)}$ almost everywhere, $\alpha = 1,\ldots,p$. Suppose further that there exist functions $g^{(\alpha)}$ such that for sufficiently small h

$$\left|\frac{F^{(\alpha)}(x+h) - F^{(\alpha)}(x)}{h}\right| \le g^{(\alpha)}(x)$$

for almost all x, and $\int_{-\infty}^{\infty} g^{(\alpha)}(x)dF^{(\alpha)}(x) < \infty$.

Then as $n_i \to \infty$, so that $n_i/N \to p_i$, $0 < p_i < 1$,

$$T_0 \xrightarrow{\mathcal{L}} \chi^2(p(k-1), \lambda_0(\phi,\underset{\sim}{\delta},F)),$$
$$T_1 \xrightarrow{\mathcal{L}} \chi^2((p-1)(k-1), \lambda_1(\phi,\underset{\sim}{\delta},F)) \qquad (21)$$
$$T_2 \xrightarrow{\mathcal{L}} \chi^2((k-1), \lambda_2(\phi,\underset{\sim}{\delta},F))$$

where

$$\lambda_0(\phi,\underset{\sim}{\delta},F) = \frac{(k-1)^2}{\mu k^2} \sum_{i=1}^{k} p_i(\underset{\sim}{\gamma}_i - \bar{\underset{\sim}{\gamma}})'\underset{\sim}{\rho}^{-1}(\underset{\sim}{\gamma}_i - \bar{\underset{\sim}{\gamma}}),$$

$$\lambda_1(\phi,\underset{\sim}{\delta},F) = \frac{(k-1)^2}{\mu k^2} \sum_{i=1}^{k} p_i(\underset{\sim}{\gamma}_i - \bar{\underset{\sim}{\gamma}})'[\underset{\sim}{\rho}^{-1} - \gamma\underset{\sim}{\rho}^{-1}\underset{\sim}{J}\underset{\sim}{\rho}^{-1}](\underset{\sim}{\gamma}_i - \bar{\underset{\sim}{\gamma}}),$$

and

$$\lambda_2(\phi,\underset{\sim}{\delta},F) = \lambda_0(\phi,\underset{\sim}{\delta},F) - \lambda_1(\phi,\underset{\sim}{\delta},F).$$

Now we are in a position to identify the sequences $\{H_N\}$ of distributions $\{\underset{\sim}{F}_N\}$ for which the criteria have limiting null distributions.

Theorem 4. Assume conditions of Theorem 3 and suppose that $q^{(\alpha)}(\phi,F)\neq 0$. Then

(i) $T_0 \xrightarrow{\mathcal{L}} \chi^2(p(k-1))$ iff H_0^* holds; furthermore, if $F^{(\alpha)} = F^{(\beta)}$ for all $\alpha \neq \beta$, then

(ii) $T_1 \xrightarrow{\mathcal{L}} \chi^2((p-1)(k-1))$ iff $\delta_i^{(1)} = \cdots = \delta_i^{(p)}$, $i=1,\ldots,k$

and

(iii) $T_2 \xrightarrow{\mathcal{L}} \chi^2(k-1)$ iff H_0^* holds, assuming $\delta_i^{(\alpha)}=\delta_i^{(\beta)}$ for all $\alpha\neq\beta$.

Remark. Note here that T_1 has a limiting central χ^2-distribution under $\{H_N\}$ only with side conditions that the form of marginal distributions is the same for all components and the location parameters are in the same relative position for each component α for the i-th population. The latter condition is similar to the statement of H_1 in the parametric case. However, here we require in addition the equality of all marginal distributions except for location parameters. This conditon is similar (in fact, equivalent) to the condition of 'commensurability' required in the parametric case (see [5]) for profile analysis to be meaningful.

Finally, in this section, we present the form of q's for some ϕ-functions referred to in section 2:

(i) $\phi = \phi_V$, $q^{(\alpha)}(\phi_V,F) = -a^{(\alpha)}(0,k-2,F)$

(ii) $\phi = \phi_B$, $q^{(\alpha)}(\phi_B,F) = a^{(\alpha)}(k-2,0,F)$

(iii) $\phi = \phi_L$, $q^{(\alpha)}(\phi_L,F) = a^{(\alpha)}(0,k-2,F) + a^{(\alpha)}(k-2,0,F)$

and

(iv) $\phi = \phi_W$, $q^{(\alpha)}(\phi_W,F) = \int_{\infty}^{\infty} f^{(\alpha)}(y)dF^{(\alpha)}(y)$.

Concluding Remarks

We have thus established, first of all, consistency of the three tests T_0, T_1, and T_2 for a specific ϕ function against alternatives to H_0^*, H_1^* and H_0^*/H_1^* respectively in the 'direction' of the specific ϕ function used. Next we have obtained their asymptotic powers for local alternatives to H_0^*, and have established that if all marginals of F are identical and the location parameters $N^{-1/2}\delta_i^{(\alpha)}$ are the same for all α, then T_1 is asymptotically $\chi^2((p-1)(k-1))$. Note from Theorem 1 that H_1^* is satisfied in such a case.

Computer programs for T_0 and T_1 have been written for specific functions ϕ_V, ϕ_B, ϕ_L and the multivariate version (see [7]) Kruskal-Wallis H-statistic. (It has been noted (see, e.g., [7]) that W-statistics (i.e., T's using ϕ_W) have the same limiting properties as H.) Also simulation studies have been carried out to investigate χ^2 approximations under H_0^* and powers under some alternatives to H_0^* (some satisfying H_1^*) for three different distributions and several covariance structures. These studies are being presented in another paper [4] and these seem to indicate that, apart from the partial justification provided for the test T_1 for the hypothesis H_1^*, there is also reasonable empirical justification to believe that indeed the concept of local alternatives to H_0^* in the direction of H_1^* might indeed provide the way out of the theoretical hurdle encountered earlier.

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